

SOME SUFFICIENT PROBLEMS FOR CERTAIN UNIVALENT FUNCTIONS

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ABSTRACT. For analytic functions $f(z)$ in the open unit disk \mathbb{U} with $f(0) = f'(0) - 1 = 0$, R. Singh and S. Singh (Coll. Math. **47**(1982), 309-314) have considered some sufficient problems for $f(z)$ to be univalent in \mathbb{U} . The object of the present paper is to discuss some sufficient problems for $f(z)$ to be some classes of analytic functions in \mathbb{U} .

1. INTRODUCTION

Let \mathcal{A} denote the class of functions $f(z)$ that are analytic in the open unit disk $\mathbb{U} = \{z \in \mathbb{C} : |z| < 1\}$, so that $f(0) = f'(0) - 1 = 0$.

We denote by \mathcal{S} the subclass of \mathcal{A} consisting of univalent functions $f(z)$ in \mathbb{U} .

Let $\mathcal{C}(\alpha)$ denote

$$\mathcal{C}(\alpha) = \{f(z) \in \mathcal{A} : |f'(z) - 1| < 1 - \alpha, \ 0 \leq \alpha < 1\}$$

and $\mathcal{C} = \mathcal{C}(0)$.

Also, let $\mathcal{S}^*(\alpha)$ be defined by

$$\mathcal{S}^*(\alpha) = \left\{ f(z) \in \mathcal{A} : \operatorname{Re} \left(\frac{zf'(z)}{f(z)} \right) > \alpha, \ 0 \leq \alpha < 1 \right\}$$

and $\mathcal{S}^* = \mathcal{S}^*(0)$.

Further, let $\mathcal{STS}(\mu)$ denote

$$\mathcal{STS}(\mu) = \left\{ f(z) \in \mathcal{A} : \operatorname{Re} \left(\frac{zf'(z)}{f(z)} \right)^{\frac{1}{\mu}} > 0, \ 0 < \mu \leq 1 \right\}$$

and $\mathcal{STS} = \mathcal{STS}(1)$.

The basic tool in proving our results is the following lemma due to Jack [1] (also, due to Miller and Mocanu [2]).

Lemma 1. *Let $w(z)$ be analytic in the open unit disk \mathbb{U} with $w(0) = 0$. Then if $|w(z)|$ attains its maximum value on the circle $|z| = r$ at a point $z_0 \in \mathbb{U}$, then we have $z_0 w'(z_0) = k w(z_0)$, where $k \geq 1$ is a real number.*

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2. CONDITIONS FOR THE CLASS \mathcal{C}

Applying Lemma 1, we drive the following result for the class \mathcal{C} .

Theorem 1. *If $f(z) \in \mathcal{A}$ satisfies*

$$|f'(z) - 1|^\beta \left| \delta + \frac{zf''(z)}{f'(z)} \right|^\gamma < \left(\frac{1+2\delta}{2} \right)^\gamma \quad (z \in \mathbb{U})$$

for some real $\beta, \gamma \geq 0$ and $\delta > -\frac{1}{2}$, then $f(z) \in \mathcal{C}$.

Proof. Let us define $w(z)$ by

$$w(z) = f'(z) - 1 \quad (z \in \mathbb{U}). \quad (1)$$

Then, clearly, $w(0) = 0$ and $w(z)$ is analytic in \mathbb{U} . Differentiating both sides in (1), we obtain

$$\frac{zf''(z)}{f'(z)} = \frac{zw'(z)}{1+w(z)},$$

and therefore,

$$|f'(z) - 1|^\beta \left| \delta + \frac{zf''(z)}{f'(z)} \right|^\gamma = |w(z)|^\beta \left| \delta + \frac{zw'(z)}{1+w(z)} \right|^\gamma < \left(\frac{1+2\delta}{2} \right)^\gamma \quad (z \in \mathbb{U}).$$

If there exists a point $z_0 \in \mathbb{U}$ such that

$$\max_{|z| \leq |z_0|} |w(z)| = |w(z_0)| = 1,$$

then Lemma 1 gives us that $w(z_0) = e^{i\theta}$ and $z_0 w'(z_0) = kw(z_0)$ ($k \geq 1$).

Thus we have

$$\begin{aligned} |f'(z_0) - 1|^\beta \left| \delta + \frac{z_0 f''(z_0)}{f'(z_0)} \right|^\gamma &= |w(z_0)|^\beta \left| \delta + \frac{z_0 w'(z_0)}{1+w(z_0)} \right|^\gamma \\ &= \left| \delta + \frac{k w(z_0)}{1+w(z_0)} \right|^\gamma \\ &= \left| \delta + \frac{k e^{i\frac{\theta}{2}}}{e^{i\frac{\theta}{2}} + e^{-i\frac{\theta}{2}}} \right|^\gamma \\ &= \left(\frac{1}{2} \right)^\gamma \left((k+2\delta)^2 + k^2 \tan^2 \frac{\theta}{2} \right)^{\frac{\gamma}{2}} \\ &\geq \left(\frac{1+2\delta}{2} \right)^\gamma. \end{aligned}$$

This contradicts our condition in the theorem. Therefore, there is no $z_0 \in \mathbb{U}$ such that $|w(z_0)| = 1$. This means that $|w(z)| < 1$ for all $z \in \mathbb{U}$. It follows that $|f'(z) - 1| < 1$ ($z \in \mathbb{U}$) so that, $f(z) \in \mathcal{C}$. \square

Letting $\beta = 1 - \lambda$, $\gamma = \lambda$ and $\delta = 1$ in Theorem 1, we have the following corollary by Singh and Singh [3].

Corollary 1. *If $f(z) \in \mathcal{A}$ satisfies*

$$|f'(z) - 1|^{1-\lambda} \left| 1 + \frac{zf''(z)}{f'(z)} \right|^\lambda < \left(\frac{3}{2} \right)^\lambda \quad (z \in \mathbb{U})$$

for some real $\lambda \geq 0$, then $f(z) \in \mathcal{C}$.

Making $\delta = 0$ in Theorem 1, we see

Corollary 2. *If $f(z) \in \mathcal{A}$ satisfies*

$$|f'(z) - 1|^\beta \left| \frac{zf''(z)}{f'(z)} \right|^\gamma < \left(\frac{1}{2} \right)^\gamma \quad (z \in \mathbb{U})$$

for some real β and $\gamma \geq 0$, then $f(z) \in \mathcal{C}$.

Remark 1. If we take $\gamma = 0$ in Corollary 2, then we have that, for some real β ,

$$|f'(z) - 1|^\beta < 1 \quad (z \in \mathbb{U})$$

implies

$$|f'(z) - 1| < 1 \quad (z \in \mathbb{U}).$$

3. CONDITIONS FOR THE CLASS $\mathcal{S}^*(\alpha)$

Next, we derive the following result for the class $\mathcal{S}^*(\alpha)$.

Theorem 2. *If $f(z) \in \mathcal{A}$ satisfies*

$$\left| \frac{zf'(z)}{f(z)} - 1 \right|^\beta \left| z \left(\frac{zf'(z)}{f(z)} \right)' \right|^\gamma < \left(\frac{1}{2} \right)^\gamma \quad (z \in \mathbb{U}) \quad (2)$$

or

$$\left| \frac{zf'(z)}{f(z)} + 1 \right|^\beta \left| z \left(\frac{zf'(z)}{f(z)} \right)' \right|^\gamma < \left(\frac{1}{2} \right)^\gamma \quad (z \in \mathbb{U}) \quad (3)$$

for some real β, γ with $\beta + 2\gamma \geq 0$, then $f(z) \in \mathcal{S}^$.*

Proof. Define $w(z)$ in \mathbb{U} by

$$G(z) = \frac{zf'(z)}{f(z)} = \frac{1+w(z)}{1-w(z)} \quad (w(z) \neq 1). \quad (4)$$

Evidently, $w(0) = 0$ and $w(z)$ is analytic in \mathbb{U} . Differentiating (4) logarithmically and simplifying, we obtain

$$\left(\frac{zf'(z)}{f(z)} \right)' = \frac{2w'(z)}{(1-w(z))^2}$$

and, hence

$$\left| \frac{zf'(z)}{f(z)} - 1 \right|^\beta \left| z \left(\frac{zf'(z)}{f(z)} \right)' \right|^\gamma = \left| \frac{2w(z)}{1-w(z)} \right|^\beta \left| \frac{2zw'(z)}{(1-w(z))^2} \right|^\gamma < \left(\frac{1}{2} \right)^\gamma \quad (z \in \mathbb{U}).$$

If there exists a point $z_0 \in \mathbb{U}$ such that

$$\max_{|z| \leq |z_0|} |w(z)| = |w(z_0)| = 1,$$

then Lemma 1 gives us that $w(z_0) = e^{i\theta}$ and $z_0 w'(z_0) = kw(z_0)$ ($k \geq 1$).

Thus we have

$$\begin{aligned} \left| \frac{z_0 f'(z_0)}{f(z_0)} - 1 \right|^\beta \left| z_0 \left(\frac{z_0 f'(z_0)}{f(z_0)} \right)' \right|^\gamma &= \left| \frac{2w(z_0)}{1-w(z_0)} \right|^\beta \left| \frac{2z_0 w'(z_0)}{(1-w(z_0))^2} \right|^\gamma \\ &= \frac{2^{\beta+\gamma} k^\gamma}{|1-w(z_0)|^{\beta+2\gamma}} \\ &\geq \left(\frac{k}{2} \right)^\gamma \geq \left(\frac{1}{2} \right)^\gamma. \end{aligned}$$

This contradicts the condition (2) in the theorem. Therefore, there is no $z_0 \in \mathbb{U}$ such that $|w(z_0)| = 1$. This means that $|w(z)| < 1$ for all $z \in \mathbb{U}$. This implies that

$$|w(z)| = \left| \frac{G(z) - 1}{G(z) + 1} \right| < 1 \quad (z \in \mathbb{U}). \quad (5)$$

It follows from (5) that

$$\operatorname{Re}(G(z)) = \operatorname{Re} \left(\frac{zf'(z)}{f(z)} \right) > 0 \quad (z \in \mathbb{U})$$

so that, $f(z) \in \mathcal{S}^*$.

Spending the same manner with (2), we conclude $f(z) \in \mathcal{S}^*$ for the condition (3). \square

Theorem 3. *If $f(z) \in \mathcal{A}$ satisfies*

$$\left| \frac{zf'(z)}{f(z)} - 1 \right|^\beta \left| z \left(\frac{zf'(z)}{f(z)} \right)' \right|^\gamma < \left(\frac{1}{2} \right)^\gamma (1-\alpha)^{\beta+\gamma} \quad (z \in \mathbb{U})$$

for some real $0 \leq \alpha < 1$, β and γ with $\beta + 2\gamma \geq 0$, then $f(z) \in \mathcal{S}^(\alpha)$.*

Proof. Defining the function $w(z)$ in \mathbb{U} by

$$G(z) = \frac{zf'(z)}{f(z)} = \frac{1 + (1-2\alpha)w(z)}{1-w(z)} \quad (w(z) \neq 1),$$

we have that $w(z)$ is analytic in \mathbb{U} and $w(0) = 0$. Since

$$\left(\frac{zf'(z)}{f(z)} \right)' = \frac{2(1-\alpha)w'(z)}{(1-w(z))^2},$$

we obtain that

$$\begin{aligned} \left| \frac{zf'(z)}{f(z)} - 1 \right|^\beta \left| z \left(\frac{zf'(z)}{f(z)} \right)' \right|^\gamma &= \left| \frac{2(1-\alpha)w(z)}{1-w(z)} \right|^\beta \left| \frac{2(1-\alpha)zw'(z)}{(1-w(z))^2} \right|^\gamma \\ &< \left(\frac{1}{2} \right)^\gamma (1-\alpha)^{\beta+\gamma} \quad (z \in \mathbb{U}). \end{aligned}$$

If there exists a point $z_0 \in \mathbb{U}$ such that

$$\max_{|z| \leq |z_0|} |w(z)| = |w(z_0)| = 1,$$

then Lemma 1 gives us that $w(z_0) = e^{i\theta}$ and $z_0 w'(z_0) = kw(z_0)$ ($k \geq 1$).

Thus we have

$$\begin{aligned} \left| \frac{z_0 f'(z_0)}{f(z_0)} - 1 \right|^\beta \left| z_0 \left(\frac{z_0 f'(z_0)}{f(z_0)} \right)' \right|^\gamma &= \left| \frac{2(1-\alpha)w(z_0)}{1-w(z_0)} \right|^\beta \left| \frac{2(1-\alpha)z_0 w'(z_0)}{(1-w(z_0))^2} \right|^\gamma \\ &= \frac{2^{\beta+\gamma} k^\gamma}{|1-w(z_0)|^{\beta+2\gamma}} (1-\alpha)^{\beta+\gamma} \\ &\geq \left(\frac{k}{2} \right)^\gamma (1-\alpha)^{\beta+\gamma} \\ &\geq \left(\frac{1}{2} \right)^\gamma (1-\alpha)^{\beta+\gamma}. \end{aligned}$$

This contradicts our condition in the theorem. Therefore, there is no $z_0 \in \mathbb{U}$ such that $|w(z_0)| = 1$. This means that $|w(z)| < 1$ for all $z \in \mathbb{U}$. This implies that

$$|w(z)| = \left| \frac{G(z) - 1}{G(z) + (1 - 2\alpha)} \right| < 1 \quad (z \in \mathbb{U}). \quad (6)$$

From (6), we obtain

$$\operatorname{Re}(G(z)) = \operatorname{Re} \left(\frac{zf'(z)}{f(z)} \right) > \alpha \quad (z \in \mathbb{U}),$$

so that, $f(z) \in \mathcal{S}^*(\alpha)$. □

4. CONDITIONS FOR THE CLASS $\mathcal{STS}(\mu)$

Using Lemma 1, we show the following result for the class $\mathcal{STS}(\mu)$.

Theorem 4. *If $f(z) \in \mathcal{A}$ satisfies*

$$\left| \frac{zf'(z)}{f(z)} \right|^\alpha \left| z \left(\frac{zf'(z)}{f(z)} \right)' \right|^\beta < \left(\frac{1}{2} \mu \right)^\beta \quad (z \in \mathbb{U})$$

for some real $\alpha \geq 0$, $\beta > 0$ and $\mu = \frac{\beta}{\alpha + \beta}$, then $f(z) \in \mathcal{STS}(\mu)$.

Proof. Letting

$$G(z) = \frac{zf'(z)}{f(z)} = \left(\frac{1+w(z)}{1-w(z)} \right)^\mu \quad (w(z) \neq 1)$$

with $\mu = \frac{\beta}{\alpha + \beta}$, we see that $w(z)$ is analytic in \mathbb{U} and $w(0) = 0$. Noting that

$$\left(\frac{zf'(z)}{f(z)} \right)' = \frac{2\mu w'(z)}{(1-w(z))^2} \left(\frac{1+w(z)}{1-w(z)} \right)^{\mu-1},$$

we have

$$\begin{aligned} \left| \frac{zf'(z)}{f(z)} \right|^\alpha \left| z \left(\frac{zf'(z)}{f(z)} \right)' \right|^\beta &= \left| \frac{1+w(z)}{1-w(z)} \right|^{\alpha\beta+\beta(\mu-1)} \left| \frac{2\mu zw'(z)}{(1-w(z))^2} \right|^\beta \\ &= \left| \frac{2\mu zw'(z)}{(1-w(z))^2} \right|^\beta < \left(\frac{1}{2}\mu \right)^\beta \quad (z \in \mathbb{U}). \end{aligned}$$

If there exists a point $z_0 \in \mathbb{U}$ such that

$$\max_{|z| \leq |z_0|} |w(z)| = |w(z_0)| = 1,$$

then Lemma 1 gives us that $w(z_0) = e^{i\theta}$ and $z_0 w'(z_0) = kw(z_0)$ ($k \geq 1$).

Thus we have

$$\begin{aligned} \left| \frac{z_0 f'(z_0)}{f(z_0)} \right|^\alpha \left| z_0 \left(\frac{z_0 f'(z_0)}{f(z_0)} \right)' \right|^\beta &= \left| \frac{2\mu z_0 w'(z_0)}{(1-w(z_0))^2} \right|^\beta \\ &= \frac{2^\beta k^\beta \mu^\beta}{|1-w(z_0)|^{2\beta}} \\ &\geq \left(\frac{k}{2}\mu \right)^\beta \geq \left(\frac{1}{2}\mu \right)^\beta, \end{aligned}$$

which contradicts our condition in the theorem. Therefore, there is no $z_0 \in \mathbb{U}$ such that $|w(z_0)| = 1$. This means that $|w(z)| < 1$ for all $z \in \mathbb{U}$. It follows that

$$G(z) = \left(\frac{1+w(z)}{1-w(z)} \right)^\mu \quad (|w(z)| < 1). \quad (7)$$

From (7), we obtain that

$$\operatorname{Re} \left(G(z)^{\frac{1}{\mu}} \right) = \operatorname{Re} \left(\left(\frac{zf'(z)}{f(z)} \right)^{\frac{1}{\mu}} \right) > 0 \quad (z \in \mathbb{U}),$$

that is, that $f(z) \in \mathcal{STS}(\mu)$. □

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